Sudip Kumar
(Assistant Professor)
Dept. of Mathematics
S. Sinha College, Aurangabad (Bihar)

B.Sc.-III

MATHEMATICS HONS: Paper-V

group B. (Multiple Integrals)

Contents:→ Line Integrals.

Remark:→ A function whose partial derivatives exists and are continuous is said to be continuously differentiable, of class C^1, or smooth.

Line Integrals:→ Let F be a vector field in the plane or space and C be a continuously differentiable path defined on the interval [a,b]. The line integral of F over C is defined by

$$\int_C F \cdot ds = \int_a^b F(c(t)) \cdot c'(t)\, dt$$

Example①:- Let $c(t) = (\sin t, \cos t, t)$, with $0 \le t \le 2\pi$, and let $F(x,y,z) = x\hat{i} + y\hat{j} + z\hat{k}$.. Compute $\int_C F \cdot ds$.

Solution:→ Here $F(c(t)) = F(\sin t, \cos t, t)$

$$= (\sin t)\hat{i} + (\cos t)\hat{j} + t\hat{k}$$

and $c'(t) = (\cos t)\hat{i} - (\sin t)\hat{j} + \hat{k}$.

$$\therefore F(c(t)) \cdot c'(t) = \sin t \cos t - \cos t \sin t + t = t$$

$$\therefore \int_C F \cdot ds = \int_0^{2\pi} t\, dt = \left[\frac{t^2}{2}\right]_0^{2\pi} = 2\pi^2$$

Ans.

Remark :→ ∵ The expression for the derivative

$$c'(t) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k}$$

$$\therefore \; c'(t)dt = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$$

The expression $ds = c'(t)\,dt$ is through of as an infinitesimal vector displacement along the curve. Writing a vector field F as $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ and taking the dot product with ds gives

$$F\cdot ds = F_1 dx + F_2 dy + F_3 dz,$$

So, we get

$$\int_C F\cdot ds = \int_C F_1 dx + F_2 dy + F_3 dz$$

We call the expression $F_1 dx + F_2 dy + F_3 dz$ a differential form.

Remark :→ ~~The~~ line integrals - Differential form Notation

The line integral of a vector field $F = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ along a path $c(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$, $a \le t \le b$ is written

$$\int_C F\cdot ds = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_a^b \left(F_1\frac{dx}{dt} + F_2\frac{dy}{dt} + F_3\frac{dz}{dt}\right) dt$$

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Example :→ Evaluate the integral $\int_C x^2 dx + xy\,dy + dz$,

Where $c : [0,1] \longrightarrow \mathbb{R}^3$ is given by

$$c(t) = (x(t), y(t), z(t)) = (t, t^2, 1).$$

Solution :→ Since $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, $\frac{dz}{dt} = 0$; therefor

$$\int_C x^2 dx + xy\,dy + dz = \cancel{\int_0^1 (t^2 + 2t^4)\,dt}$$

$$= \int_0^1 \left([x(t)]^2 \cdot \frac{dx}{dt} + [x(t) y(t)] \frac{dy}{dt} \right) dt$$

$$= \int_0^1 (t^2 + 2t^4)\,dt = \left[\frac{1}{3} t^3 + \frac{2}{5} t^5 \right]_0^1 = \frac{11}{15} \quad \underline{\text{Ans.}}$$

Example 3 :→ Evaluate the integral $\int_C \cos z\,dx + e^x dy + e^y dz$,

Where $c(t) = (1, t, e^t)$ and $0 \le t \le 2$.

Solution :→ $\because \frac{dx}{dt} = 0$, $\frac{dy}{dt} = 1$, $\frac{dz}{dt} = e^t$ & so,

$$\int_C \cos z\,dx + e^x dy + e^y dz = \int_0^2 (0 + e + e^t \cdot e^t)\,dt$$

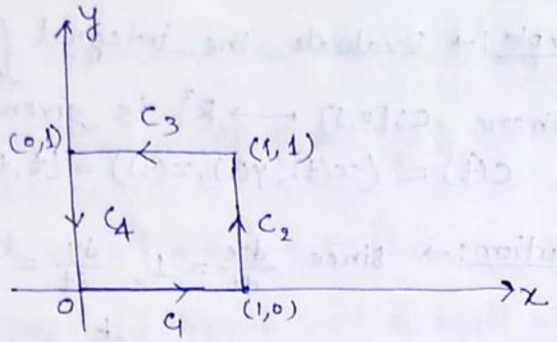
$$= \int_0^2 (e + e^{2t})\,dt = \left[et + \frac{1}{2} e^{2t} \right]_0^2$$

$$= 2e + \frac{1}{2} e^4 - \frac{1}{2}$$

<u>Ans.</u>

Example ④ :→ Let C be the perimeter of the unit square $[0,1] \times [0,1]$ in the plane, traversed in the counter clockwise direction. Evaluate the line integral $\int_C F \cdot ds$, Where $F(x,y) = x^2 \hat{i} + xy \hat{j}$

Solution: →



The parametrizations are

$C_1 : (t, 0), \ 0 \le t \le 1; \ c_1'(t) = \hat{i} ; F(c_1(t)) = t^2 \hat{i}$

$C_2 : (1, t), \ 0 \le t \le 1; \ c_2'(t) = \hat{j} ; F(c_2(t)) = \hat{i} + t\hat{j}$

$C_3 : (1-t, 1), \ 0 \le t \le 1; \ c_3'(t) = -\hat{i} ; F(c_3(t)) = (1-t)^2 \hat{i} + (1-t)\hat{j}$

$C_4 : (0, 1-t), \ 0 \le t \le 1, \ c_4'(t) = -\hat{j} ; F(c_4(t)) = 0.$

Thus.

$$\int_{C_1} F \cdot ds = \int_0^1 t^2 \, dt = \frac{1}{3}$$

$$\int_{C_2} F \cdot ds = \int_0^1 t \, dt = \frac{1}{2}$$

$$\int_{C_3} F \cdot ds = \int_0^1 (1-t)^2 (-1) \, dt = -\frac{1}{3}$$

$$\int_{C_4} F \cdot ds = \int_0^1 0 \cdot dt = 0$$

Adding, we get

$$\int_C F \cdot ds = \frac{1}{3} + \frac{1}{2} - \frac{1}{3} + 0 = \frac{1}{2}$$ #